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On the Approximation of Fourier Series by Abel Means

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This paper deals with the pointwise saturation problem for the Abel means of Fourier series. Most of the known results on pointwise approximation in connection with summation processes of Fourier series are Jackson-type theorems. (See the books of N. Achieser, P. P. Korovkin, and I. P. Natanson on approximation theory. We wish also to refer to the results due to G. Alexits and his school, cf. [1] and [5]). Here we shall study a Bernstein-type problem for the Abel means, i.e., a converse of the Jackson-type problems. But we shall study it only in case of saturated approximation. Although this type of question has been studied for several years, only a few results are available. Bajšanski–Bojanić [3] proved a pointwise "o"-theorem for the Bernstein polynomials, and recently V. A. Andrienko [2] studied this problem for the Fejér means of Fourier series. A generalization of his result was given by the author [4]. On the other hand, these problems are closely connected with theorems concerning generalized derivatives of functions which have their origin in Schwarz's theorem [9, p. 431] and its generalizations due to C. de la Vallée Poussin (see [7] for more recent results).

Throughout this paper we shall deal with the space $L_{2\pi}$ of Lebesgue integrable real-valued functions $f(x)$ which are periodic with period 2π. Let f be in $L_{2\pi}$; we denote

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} A_k(x), \qquad A_k(x) = a_k \cos kx + b_k \sin kx,$$

where a_k , b_k are the Fourier coefficients of f. Its conjugate series is defined by

$$f^\sim(x) \sim \sum_{k=1}^{\infty} B_k(x), \qquad B_k(x) = a_k \sin kx - b_k \cos kx.$$

The Abel means of $f(x)$ and $f^\sim(x)$ are given, respectively, by

$$f(r, x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} r^k A_k(x) \quad \text{and} \quad f^\sim(r, x) = \sum_{k=1}^{\infty} r^k B_k(x), \qquad 0 \leqslant r \leqslant 1.$$

The saturation theorem for the Abel means states the following. Let $X_{2\pi}$ be one of the spaces $C_{2\pi}$ or $L^p_{2\pi}$, $1 \leqslant p < \infty$.

(a) If f, g in $X_{2\pi}$ are such that

$$\lim_{r\to 1^-} \left\| \frac{f(r,\cdot)-f(\cdot)}{1-r} - g(\cdot) \right\|_{X_{2\pi}} = 0, \tag{1}$$

then $f^\sim$ is absolutely continuous and $f^{\sim\prime} = -g$, and vice versa. If, in particular, $g = 0$, i.e., $\|f(r,\cdot) - f(\cdot)\|_{X_{2\pi}} = o(1-r)$ as $r \to 1^-$, then $f = \text{const}$.

(b) For an f in $X_{2\pi}$ the following statements are equivalent:

(i) $\|f(r,\cdot) - f(\cdot)\|_{X_{2\pi}} = O(1-r) \qquad (r \to 1^-)$.

(ii) $f^\sim \in \mathrm{Lip}(1, X_{2\pi})$.

Part (a) of the theorem goes back to E. Hille, while part (b) is due to Butzer for $L^p_{2\pi}$-spaces, $1 < p < \infty$, and to Sunouchi–Watari for the spaces $C_{2\pi}$ and $L_{2\pi}$ (see [6, p. 118]).

What can be said about the function $f(x)$ if the limit

$$\lim_{r\to 1} (f(r,x) - f(x))/(1-r) = g(x)$$

exists pointwise? In case $g(x) = 0$, we have the so-called pointwise "o"-theorem.

The main result, Theorem 1, is stated and proved in the next section, while in the final section we study its major consequences.

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THEOREM 1. *Let $f \in L_{2\pi}$ be finite in some interval (a, b) and such that $\lim_{r\to 1^-} f(r,x) = f(x)$ for all $x \in (a, b)$. If*

$$\lim_{r\to 1^-} (f(r,x) - f(x))/(1-r) = g(x)$$

exists finitely for all $x \in (a, b)$, with $g(x)$ integrable, then for almost all x in (a, b),

$$f^\sim(x) = C - \int_a^x g(u)\,du, \tag{2}$$

where C is some constant.

Remark 1. In this form, Theorem 1 is closely related to the following theorem of de la Valleé Poussin:

Let $F(x)$ be continuous in $[a, b]$, finite, and such that

$$\lim_{t\to 0^+} (F(x - t) - 2F(x) + F(x - t))/t^2 = g(x)$$

exists finitely for all $x \in (a, b)$, with $g(x)$ integrable. Then

$$F(x) = C_1 x + C_2 + \int_a^x du_1 \int_a^{u_1} g(u_2)\, du_2 ,$$

where C_1 and C_2 are some constants.

We shall prove Theorem 1 by reducing it to a generalized version of this result.

In the proof of Theorem 1 we shall use three lemmas which are of interest in their own right. Lemma 3 may be considered a generalization of Rajchman's lemma (cf. [10, p. 353]).

LEMMA 1. *Let* $f \in L_{2\pi}$. *For every* x *for which* $\lim_{r\to 1^-} f(r, x) = c(x)$ *exists finitely, we have*

$$\liminf_{\delta\to 0^+} \frac{1}{4\pi} \int_\delta^\pi \frac{\varphi(x, u)}{\sin^2 u/2}\, du \leqslant \lim_{r\to 1^-} \frac{f(r, x) - c(x)}{1 - r}$$

$$\leqslant \limsup_{\delta\to 0^+} \frac{1}{4\pi} \int_\delta^\pi \frac{\varphi(x, u)}{\sin^2 u/2}\, du, \qquad (3)$$

where $\varphi(x, u) = f(x + u) - 2c(x) + f(x - u)$. *In particular,* (3) *holds for almost all* x, *with* $c(x)$ *replaced by* $f(x)$.

Proof. Let x be a point for which $\lim_{r\to 1^-} f(r, x) = c(x)$ exists finitely. For brevity, we shall denote by $[\underline{D}_{\{1\}} f](x)$ and $[\bar{D}_{\{1\}} f](x)$ the extreme left and the extreme right side of (3), respectively. We shall only verify the right inequality of (3).

If $[\bar{D}_{\{1\}} f](x) = +\infty$, there is nothing to prove. Suppose $d = [\bar{D}_{\{1\}} f](x)$ is finite. Given $\epsilon > 0$, there exists a $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$\frac{1}{4\pi} \int_\delta^\pi \frac{\varphi(x, u)}{\sin^2 u/2}\, du < d + \epsilon$$

for $0 < \delta < \delta_0$. On the other hand,

$$\frac{f(r, x) - c(x)}{1 - r} = \frac{1}{2\pi} \int_0^\pi \varphi(x, u) \frac{1 + r}{1 - 2r\cos u + r^2}\, du$$

$$= \frac{1 + r}{4\pi} \left\{ \int_0^{\delta_0} + \int_{\delta_0}^\pi \right\} \frac{\varphi(x, u)}{\sin^2 u/2} \frac{1 - \cos u}{1 - 2r\cos u + r^2}\, du$$

$$= I_1(r) + I_2(r).$$

Since the function

$$p_r(u) = \frac{(1+r)(1-\cos u)}{1-2r\cos u + r^2} \qquad (0 \leqslant r < 1) \tag{4}$$

is monotonically increasing on $0 \leqslant u \leqslant \pi$, with $p_r(0) = 0$ and $p_r(\pi) = 2/(1+r)$, we obtain by the second law of means for integrals

$$I_1(r) = \frac{p_r(\delta_0)}{4\pi} \int_{\delta_r}^{\delta_0} \frac{\varphi(x,u)}{\sin^2 u/2}\, du$$

$$< p_r(\delta_0) \left\{ d + \epsilon - \frac{1}{4\pi} \int_{\delta_0}^{\pi} \frac{\varphi(x,u)}{\sin^2 u/2}\, du \right\}, \qquad (0 < \delta_r < \delta_0).$$

Moreover, since as $r \to 1^- \, p_r(u)$ converges to 1 uniformly in $0 < \delta_0 \leqslant u \leqslant \pi$, we get

$$\limsup_{r\to 1^-} I_1(r) \leqslant d + \epsilon - \frac{1}{4\pi} \int_{\delta_0}^{\pi} \frac{\varphi(x,u)}{\sin^2 u/2}\, du$$

and, for the second integral,

$$\lim_{r\to 1^-} I_2(r) = \frac{1}{4\pi} \int_{\delta_0}^{\pi} \frac{\varphi(x,u)}{\sin^2 u/2}\, du.$$

Hence

$$\limsup_{r\to 1^-} \frac{f(r,x) - c(x)}{1-r} \leqslant d + \epsilon,$$

for every $\epsilon > 0$.

In case $[\bar{D}_{\{1\}} f](x) = -\infty$, one proves that

$$\limsup_{r\to 1^-} \frac{f(r,x) - c(x)}{1-r} \leqslant K$$

for every real constant K. This proves the right part of inequality (3).

Finally, it is well known that almost everywhere $\lim_{r\to 1^-} f(r,x)$ exists finitely and equals $f(x)$; this takes care of the last statement of the lemma.

LEMMA 2. *Let $f \in L_{2\pi}$, and let*

$$g(r,x) = -\sum_{k=1}^{\infty} r^k k A_k(x) \qquad (0 \leqslant r < 1), \tag{5}$$

i.e., $g(r,x) = -f^{\sim\prime}(r,x)$. For any fixed x for which $\lim_{r\to 1^-} f(r,x) = c(x)$ exists finitely, we have

$$\varliminf_{r\to 1^-} g(r,x) \leqslant \varliminf_{r\to 1^-} \frac{f(r,x) - c(x)}{1-r} \leqslant \limsup_{r\to 1^-} g(r,x).$$

Proof. By definition of the function $g(r, x)$,

$$f(r, x) - c(x) = \int_r^1 g(\rho, x) \frac{d\rho}{\rho},$$

whenever it is meaningful. The rest of the proof is clear.

LEMMA 3. *Let $f \in L_{2\pi}$, and let*

$$F(x) = C + \sum_{k=1}^{\infty} A_k(x)/k, \tag{6}$$

whenever the sum converges. If $\lim_{r\to 1^-} f(r, x_0) = c(x_0)$ exists finitely, then

$$[\underline{D}^2 F](x_0) \leqslant \limsup_{r\to 1^-} \frac{f(r, x_0) - c(x_0)}{1 - r} \tag{7a}$$

and

$$\liminf_{r\to 1^-} \frac{f(r, x_0) - c(x_0)}{1 - r} \leqslant [\bar{D}^2 F](x_0), \tag{7b}$$

where $[\underline{D}^2 F](x_0)$ and $[\bar{D}^2 F](x_0)$ are, respectively, the lower and upper second Riemann derivative of $F(x)$ at $x = x_0$.

By Tauber's first theorem (cf. [8, p. 149]), series (6) converges whenever $\lim_{r\to 1^-} F(r, x)$ exists finitely, in particular, for all x for which $\lim_{r\to 1^-} f(r, x)$ exists finitely. Moreover, since by the saturation theorem, part (a),

$$F(x) = C + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - u) \log \frac{1}{4 \sin^2 u/2}\, du \qquad \text{a.e.},$$

and since the logarithmic function under the integral sign belongs to every $L_{2\pi}^p$, $1 \leqslant p < \infty$, $F(x)$ is continuous whenever f belongs to some $L_{2\pi}^p$, $1 < p \leqslant \infty$.

Remark 2. By replacing $\{f(r, x_0) - c(x_0)\}/(1 - r)$ by $g(r, x)$ in (7a) and (7b), we obtain Rajchman's lemma. However, Rajchman proved his lemma under the weaker condition that the function $F(x)$ defined in (6) is a Fourier series which is Abel summable at $x = x_0$, while in our case the associated series $\sum_1^\infty A_k(x)$ is already a Fourier series which is Abel summable at $x = x_0$.

Proof. It is enough to prove one of the inequalities (7), say (7a). Without loss of generality we may suppose that $\int_{-\pi}^{\pi} f(x)\, dx = 0$, $x_0 = 0$, and $F(x)$ is even. The desired inequality will then follow, if we can prove that, for any m, $[\underline{D}^2 F](0) > m$ implies

$$\limsup_{r\to 1^-} \frac{1}{\log 1/r} \int_r^1 g(\rho; 0) \frac{d\rho}{\rho} \geqslant m,$$

($\log(1/r) \sim 1 - r$ as $r \to 1^-$). Replacing $F(x)$ by $F(x) - m(1 - \cos x)$, we may further assume that $m = 0$.

Suppose that $[\underline{D}^2F](0) > 0$ and that, contrary to what we want to prove,

$$\limsup_{r\to 1^-} \frac{1}{\log 1/r} \int_r^1 g(\rho, 0) \frac{d\rho}{\rho} < 0. \tag{8}$$

(The argument up to this point is Rajchman.) With $G(r) = F(r, 0)$, $rG'(r) = f(r, 0)$, assumption (8) implies that there are $\epsilon > 0$ and $0 < r_0 < 1$ such that

$$rG'(r) - G'(1) < -\epsilon \log 1/r, \qquad r_0 < r < 1,$$

where $G'(1) = \lim_{r\to 1^-} f(r, 0) = c(0)$. Moreover,

$$\begin{aligned} G(1) - G(r) - \log(1/r)\, G'(1) &= \int_r^1 \{\rho G'(\rho) - G'(1)\} \frac{d\rho}{\rho} \\ &< -\frac{\epsilon}{2} \log^2(1/r), \qquad r_0 < r < 1, \end{aligned}$$

or

$$0 < \liminf_{r\to 1^-} (2/\log^2 1/r)\{G(r) - G(1) + \log(1/r)\, G'(1)\}.$$

We shall prove that this inequality is false. Indeed, we prove that

$$\limsup_{r\to 1^-} (2/\log^2 1/r)\{G(r) - G(1) + \log(1/r)\, G'(1)\} < 0.$$

By Lemma 2 we obtain immediately

$$\lim_{r\to 1^-} \frac{G(r) - G(1)}{\log 1/r} = -G'(1).$$

Furthermore, a moment's reflection shows that, by our hypothesis on $F(x)$ at $x = 0$,

$$\lim_{\delta\to 0^+} \frac{1}{4\pi} \int_\delta^\pi \frac{\Phi(u)}{\sin^2 u/2}\, du = -G'(1)$$

by virtue of Lemma 1 and the previous formula. Here,

$$\Phi(u) = F(u) - 2F(0) + F(-u) = 2F(u).$$

Thus, for each $0 < r < 1$,

$$\begin{aligned} I(r) &= (2/\log^2 1/r)\{G(r) - G(1) + \log(1/r)\, G'(1)\} \\ &= \frac{2}{\log 1/r} \lim_{\delta\to 0^+} \left\{ \frac{1-r}{\log 1/r} \frac{1}{4\pi} \int_\delta^\pi \frac{\Phi(u)}{\sin^2 u/2} p_r(u)\, du - \frac{1}{4\pi} \int_\delta^\pi \frac{\Phi(u)}{\sin^2 u/2}\, du \right\}, \end{aligned}$$

where the function $p_r(u)$ is given by (4). Setting

$$q_r(u) = \frac{2}{1+r} - p_r(u) = \frac{(1-r)^2(1+\cos u)}{(1+r)(1-2r\cos u + r^2)}$$

($q_r(u)$ is nonnegative and $O[(1-r)^2]$ as $r \to 1^-$, uniformly in u in any interval $0 < \delta_0 \leqslant u \leqslant \pi$), we have

$$I(r) = -\frac{2(1-r)}{\log^2 1/r} \lim_{\delta \to 0^+} \frac{1}{4\pi} \int_\delta^\pi \frac{\Phi(u)}{\sin^2 u/2} q_r(u)\, du + \frac{2}{\log 1/r}\left(\frac{1-r}{\log 1/r}\frac{2}{1+r} - 1\right) G'(1).$$

The hypothesis $[\underline{D}^2 F](0) > 0$ implies that, for some $\epsilon > 0$ and $\delta_0 = \delta_0(\epsilon) > 0$, $\Phi(u)/\sin^2 u/2 > \epsilon$ for all $0 < u < \delta_0$. Hence

$$\frac{1}{4\pi}\int_\delta^{\delta_0} \frac{\Phi(u)}{\sin^2 u/2} q_r(u)\, du > \frac{\epsilon}{4\pi}\int_\delta^{\delta_0} q_r(u)\, du \qquad (0 < \delta < \delta_0)$$

and, consequently,

$$I(r) < -\frac{2(1-r)}{\log^2 1/r}\left\{\frac{\epsilon}{4\pi}\int_0^{\delta_0} q_r(u)\, du + \frac{1}{4\pi}\int_{\delta_0}^\pi \frac{\Phi(u)}{\sin^2 u/2} q_r(u)\, du\right\} + \frac{2}{\log 1/r}\left(\frac{1-r}{\log 1/r}\frac{2}{1-r} - 1\right) G'(1).$$

Since $(1/4\pi)\int_0^\pi q_r(u)\, du = (1-r)/4(1+r)$ and since

$$\lim_{r \to 1^-} \frac{2}{\log 1/r}\left(\frac{1-r}{\log 1/r}\frac{2}{1-r} - 1\right) = 0,$$

we obtain by Lebesgue's dominated convergence theorem that

$$\limsup_{r \to 1^-} I(r) \leqslant -\epsilon/4 < 0,$$

which proves the lemma.

Proof of Theorem 1. Since $\lim_{r\to 1^-} f(r, x) = f(x)$ for all x in (a, b), $F(x)$ is well-defined in (a, b) (see Remark 1). If, in particular, $F(x)$ is continuous in (a, b), then the theorem is an immediate consequence of Lemma 3 and the following result of de la Vallée Poussin [10, p. 327]:

Let $F(x)$ be a continuous function on some interval $a < x < b$, and let $g(x)$ be finite valued and integrable on the same interval. If, for each x,

$$[\underline{D}^2 F](x) \leqslant g(x) \leqslant [\bar{D}_2 F](x), \tag{9}$$

then

$$F(x) = \int_a^x du_1 \int_a^{u_1} g(u_2)\, du_2 + C_1 x + C_2\,, \tag{10}$$

where C_1 and C_2 are constants.

Indeed, by Fatou's theorem [10, p. 99], (10) implies (2).

It remains to prove the continuity of $F(x)$ in (a, b). This, however, follows by the same arguments used in proving Verblunsky's uniqueness theorem for Abel summable trigonometric series, as given in [10, p. 355].

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A consequence of Theorem 1 is the following pointwise analog of part (a) of the saturation theorem.

THEOREM 2. *Let $f(x)$ be a finite-valued function in $L_{2\pi}$ such that $\lim_{r\to 1^-} f(r, x) = f(x)$ for all x. If the limit*

$$\lim_{r\to 1^-} \frac{f(r, x) - f(x)}{1 - r} = g(x) \tag{11}$$

exists for all x, with $g(x)$ integrable, then for almost all x

$$f^{\sim}(x) = C - \int_{-\pi}^{x} g(u)\, du,$$

where

$$C = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \int_{-\pi}^{x} g(u)\, du.$$

If, in particular, $g(x) = 0$ for all x, then $f(x)$ is a constant.

It is quite simple to prove that relation (1) in the saturation theorem is equivalent to

$$\lim_{r\to 1^-} \| g(r, \cdot) - g(\cdot) \|_{X_{2\pi}} = 0,$$

$g(r, x)$ being given by (5). If we replace (11) by the stronger condition

$$\lim_{r\to 1^-} g(r, x) = g(x),$$

then Theorem 2 reduces to Verblunsky's theorem, which states that the trigonometric series

$$-\sum_{k=1}^{\infty} k A_k(x) \tag{12}$$

is the Fourier series of g. However, Verblunsky proved his result under the condition that the coefficients of (12) are $o(k)$, which is trivially satisfied in our case (see [10, p. 352 ff.]).

COROLLARY 1. *For continuous functions* $f(x)$, *Theorem* 2 *remains true if* (11) *is satisfied for all* x *except a denumerable set of points.*

Proof. In this case, the associated function $F(x)$ of $f(x)$ is smooth ($F \in \mathrm{Lip}^*(1, C_{2\pi})$) and, for smooth functions, de la Vallée Poussin's result remains true even if (9) is violated in a denumerable set of points.

Finally we obtain, as an immediate consequence of the saturation theorem and Lemma 2, the following theorem.

THEOREM 3. *Let* $f \in L_{2\pi}$. *If* $f^{\sim}$ *is absolutely continuous, then for almost all* x,

$$\lim_{r \to 1^-} \frac{f(r, x) - f(x)}{1 - r} = -f^{\sim\prime}(x).$$

More generally, if $f^{\sim} \in BV_{2\pi}$, *then*

$$\lim_{r \to 1^-} \frac{f(r, x) - f(x)}{1 - r}$$

exists almost everywhere.

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